

# New results in relation with second-order constraint qualification

C. Eugenio Echague, M. Laura Schuverdt

*Departamento de Matemática*

*Facultad de Ciencias Exactas, Universidad Nacional de La Plata, Argentina.*

January 31, 2009

The study of Necessary Optimality Conditions (NOC) is quite relevant for practical Nonlinear Programming. In optimization one is interested in finding global minimizers. Since this is very hard, most practical algorithms find only stationary points, that is, points that satisfy some NOC. Necessary Optimality Conditions may be first-order or second-order, according to the use of derivatives in its formulation.

A first order constraint qualification (CQ1) is a property of the feasible points such that, when it is satisfied by a local minimizer, it guarantees that the Karush-Kuhn-Tucker (KKT) conditions takes place at that point. Then, the proposition

$$KKT \text{ or not } CQ1 \tag{1}$$

is a first-order NOC. In fact, in the convergence analysis of many algorithms the authors prove that its limit points satisfy a NOC like (1). It is well known that conditions like Regularity [4], Mangasarian-Fromovitz [4], Constant Rank [7], Constant Positive Linear Dependence [1], Quasi-Normality [5] and Abadie [6] are first-order constraint qualifications. The next tree shows the relationship between the first-order quality constraints exposed in the last paragraph

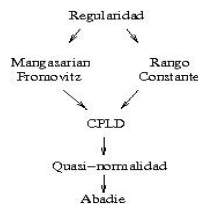


Figure 1: First-order constraint qualifications and their implications

A second-order constraint qualification (CQ2) is a property of the feasible KKT points of a nonlinear programming problem such that, when it is satisfied by a local minimizer, it guarantees that the Hessian of the Lagrangian function associated to, at least one vector of Lagrange multipliers is positive semi-definite in a tangent space (SDP). Then, for KKT points, the proposition

$$SDP \text{ or not } CQ2 \tag{2}$$

is a second-order NOC. It is well known that the Regularity is a second order constraint qualification [4]. It is also known that Mangasarian-Fromovitz is not a second-order NOC [3]. Then no other first-order constraint qualification below Mangasarian-Fromovitz in the tree (1) is a second order constraint qualification.

It was an open question if Constant Rank is a second-order constraint qualification. In this work, we prove that it is. Moreover, we prove that at a local minimizer, for any vector of Lagrange Multipliers, the Hessian of the Lagrangian function is positive semi-definite in the biggest tangent space found in the literature. We also prove that any first-order constraint qualification in addition to the Weak Constant Rank property introduced by Andreani et al. in [2], is a second-order constraint qualification. In fact, we prove that at a local minimizer, for any vector of Lagrange multipliers, the Hessian of the Lagrangian Function is positive semi-definite in the reduced tangent space found in the literature.

## References

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