A CHAOTIC ALGORITHM FOR CLOSED-LOOP STOCHASTIC OPTIMIZATION PROBLEMS

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In this presentation we deal with stochastic optimization problems. These problems can be split into two categories. First, the *open-loop* problems where the decision variables are deterministic. Numerical solution of such problems needs an approximation step of the objective expectation, this can be done through a *stochastic gradient* algorithm. Second, *closed-loop* problems where the decision variable is a function of the noise or observation random variable. Numerical solution of such problems needs *two* approximation steps: the approximation of the objective expectation and the approximation of the information structure represented by the functional decision variable. Standard stochastic gradient algorithms can not be applied to this kind of problems.

In this paper we are concerned with a modified stochastic gradient algorithm, made to achieve tractable solution methods for closed-loop stochastic optimization problems.

The stochastic algorithms literature was traditionally concerned with the solution of open-loop stochastic optimization problems. This kind of algorithms consists on a mix between classical deterministic projected gradient algorithms and Monte-Carlo sampling methods, where we approximate at each step the gradient of the cost function J with an independent sampling.

In the closed-loop case, we are concerned with solutions of problems of the following type (the control u is now a function of the noise $\boldsymbol{\xi}$):

$$\min_{u \in \mathcal{N}^{\mathrm{me}} \cap \mathcal{N}^{\mathrm{cv}}} J(u) \stackrel{\mathrm{def}}{=} \mathbb{E}\left[j\left(u(\boldsymbol{\xi}), \boldsymbol{\xi}\right)\right], \quad (1)$$

where $\mathcal{N} = L^2(\mathbb{R}^n, \mathbb{B}(\mathbb{R}^n), \mathbb{P}_{\boldsymbol{\xi}})$ is a functional Hilbert space[1], $\mathcal{N}^{\mathrm{me}} \subset \mathcal{N}$ is a ν-dimensional Hilbert subspace (mainly a measurability - recourse constraints which can be written as $\{u \in \mathcal{N}, \quad u \preceq h\}$ where h is a measurable mapping[2]) and $\mathcal{N}^{\mathrm{cv}} \subset \mathcal{N}$ is a feasible closed and convex subset.

The literature concerning stochastic algorithms within the closed loop framework appears to be more restricted. Nevertheless, we can cite the recent works of Barty et al. concerned with adapting the stochastic gradient algorithm to the closed loop case [1, 2, 3]. It consists on the approximation of the functional gradient by a convolution product with a kernel function $\mathcal{K} : \Xi \to \mathbb{R}$ such that $\mathcal{K} \preceq h$.

A more conventional method present in the literature is the *decision rules* method which is often used to solve closed loop stochastic optimal control problems. In this method we look for a sub-optimal solution in a subspace generated by a finite set of decision rules or feedback functions. For this method, we refer to the works of S.J. Garstka and R.J.B. Wets [4], of A. Shapiro and A. Nemirovski [5], and recently the works of J.P. Vial and J. Thénié [6, 7, 8].

Our proposed algorithm is meant to link these two methods: it's a stochastic gradient algorithm for closed loop problems using a probabilistic decision rules structure in order to approximate the optimal feedbacks.

The subspace $\mathcal{N}^{\mathrm{me}}$ being an Hilbert space, we denote by $(\phi_i)_{i\in I}$ a basis of that space such that $\mathrm{card}(I) = \dim(\mathcal{N}^{\mathrm{me}})$. Therefore, we can propose the following algorithm:

Algorithm 1 (Chaotic stochastic gradient algorithm for (1))**.**

- **Step [0] :**
  - Let $u^{[0]} \in \mathcal{N}^{\mathrm{me}} \cap \mathcal{N}^{\mathrm{cv}}$ be an initial control.
  - Let $\mathbb{P}_{\boldsymbol{i}} \sim (p_i)_{i\in I}$ be a probability law over $I \subset \mathbb{N}$.
- **Step [k] :**

[1] $\mathbb{B}(\mathbb{R}^n)$ is the Borel σ-field of $\mathbb{R}^n$ and $\mathbb{P}_{\boldsymbol{\xi}}$ is the probability low of the random variable $\boldsymbol{\xi}$

[2] the operator $\preceq$ means measurable with respect to

– Let $\boldsymbol{i}^{[k+1]} \in I$ be a random variable independent from $(\boldsymbol{i}^{[1]}, \dots, \boldsymbol{i}^{[k]})$ following the probability law $\mathbb{P}_{\boldsymbol{i}}$,
– Let $\boldsymbol{\xi}^{[k+1]}$ be a random variable independent from $(\boldsymbol{\xi}^{[1]}, \dots, \boldsymbol{\xi}^{[k]})$ following the probability law $\mathbb{P}_{\boldsymbol{\xi}}$.

$$\boldsymbol{u}^{[k+1]}(\cdot) = \Pi_{\mathcal{N}^{\text{cv}}} \left(\boldsymbol{u}^{[k]}(\cdot) - \rho^{[k]} \left\langle \phi_{\boldsymbol{i}^{[k+1]}}(\boldsymbol{\xi}^{[k+1]}), \nabla_u j \left(\boldsymbol{u}^{[k]}(\boldsymbol{\xi}^{[k+1]}), \boldsymbol{\xi}^{[k+1]} \right) \right\rangle_{\mathbb{U}} \frac{\phi_{\boldsymbol{i}^{[k+1]}}(\cdot)}{p_{\boldsymbol{i}^{[k+1]}}} \right),$$

where $\Pi_{\mathcal{N}^{\text{cv}}}$ is the projection over the closed convex subset $\mathcal{N}^{\text{cv}}$.

The standard decision rule method described previously has two major drawbacks. In one hand, the a priori choice of a basis functions introduces an incompressible bias which can be reduced only by adding new functions into the basis. In the other hand, for a given number of basis function the mean square error introduced by the estimator of the optimal solution depends on the number of Monte-Carlo samples: the variance reduces as the number of samples increases.

In contrary, our chaotic stochastic gradient algorithm perform the optimization step along with approximating the information structure (by decision rules) and approximating the expectation of the cost function (Monte-Carlo). Virtually we can have as many functions as we want (it suffices that we increase the cardinality of I) and as many Monte-Carlo samples as we want (by increasing the algorithm's steps).

In this presentation we present two different results related to algorithm (1): a convergence theorem and a central limit theorem. We also illustrate our results by presenting a numerical application of our algorithm applied to a management problem of an hydro-electric dam.

References

1. K. Barty, J.-S. Roy, and C. Strugarek, *A perturbed gradient algorithm in Hilbert spaces*, http://www.optimization-o nline.org, 2005.
2. ———, *A stochastic gradient type algorithm for closed loop problems*, submitted, available on-line at http://dochost.rz.hu-berlin.de/speps/, 2005.
3. ———, *Temporal difference learning with kernels for pricing american-sty le options*, http://www.optimization-online.org/DB_HTML/2005/05/1133.html, 2005.
4. S.J. Garstka and R.J.B. Wets, *On decision rules stochastic programming*, Mathematical Programming **7** (1974), 117–173.
5. A. Shapiro and A. Nemirovski, *On complexity of stochastic programming problems*, e-print in optimization online.
6. J. Thénié and J.P. Vial, *Programmation stochastique avec règles de décision linéaires*, 10th International Conference on Stochastic Programming, Tucson, AZ, USA, 9-15 Octobre 2004.
7. ———, *Programmation stochastique avec règles de décision linéaires*, Présentation dans ROADEF'05, Tours, France, 14-16 Fé vrier 2005.
8. ———, *Step decision rules for multistage stochastic programmi ng: a heuristic approach*, http://www.optimization-online.org/DB_HTML/2006/08/1440.html, 2006.

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