

# Extensions of Incomplete Oblique Projections Method for Solving Large-Scale Non-Negativity Constrained Least Squares Problems \*

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## Abstract

The aim of this paper is to extend the applicability of an algorithm for solving inconsistent linear systems to the non-negativity constrained case, by employing incomplete projections onto the set of solutions of the augmented system  $Ax - r = b$ , together with the non-negativity constraints. The extended algorithm converges to the minimal norm solution of the non-negativity constrained least squares solutions. The incomplete oblique projections used are defined by means of matrices that penalize the norm of the residuals.

Large and sparse systems of linear equations arise in many important applications [1, 2, 7], as image reconstruction from projections, radiation therapy treatments planning, computational mechanics and optimization problems. In practice, those systems are often inconsistent, and one usually seeks a point  $x^* \in \mathbb{R}^n$ ,  $x^* \geq 0$ , that minimizes a certain proximity function.

In [6] we have introduced for inconsistent problems the IOP algorithm that converges to a weighted least squares solution of the system  $Ax = b$ . This algorithm uses an incomplete oblique projections scheme onto the solution set of the augmented system  $Ax - r = b$ . More explicitly, in order to solve a possibly inconsistent system  $Ax = b$ ,  $A \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^m$ , we consider the standard problem:

$$\min_{x \in \mathbb{R}^n} \|b - Ax\|_{D_m}^2, \quad (1)$$

where  $\|\cdot\|_{D_m}$  denotes the norm induced by the positive definite diagonal matrix  $D_m \in \mathbb{R}^{m \times m}$ . In [6] we proved that this is equivalent to the problem

$$\min\{\|p - q\|_D^2 : \text{for all } p \in \mathcal{P} \text{ and } q \in \mathcal{Q}\},$$

$\mathcal{P}$  and  $\mathcal{Q}$  being two convex sets in the  $(n + m)$ -dimensional space  $\mathbb{R}^{n+m}$ , such that denoting by  $[u; v]$  the vertical concatenation of  $u \in \mathbb{R}^n$ , with  $v \in \mathbb{R}^m$ ,

$$\mathcal{P} = \{p : p = [x; r] \in \mathbb{R}^{n+m}, \quad x \in \mathbb{R}^n, \quad r \in \mathbb{R}^m, \quad Ax - r = b\}, \text{ and}$$

$$\mathcal{Q} = \{q : q = [x; 0] \in \mathbb{R}^{n+m}, \quad x \in \mathbb{R}^n, 0 \in \mathbb{R}^m\},$$

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adopting the distance  $d(p, q) = \|p - q\|_D$ , for all  $p \in \mathcal{P}$ ,  $q \in \mathcal{Q}$ .  $D$  is a diagonal matrix of order  $n + m$ , whose  $n$  first elements are 1's, and the last  $m$  coincide with those of  $D_m$ . That result led us to develop the IOP method for solving (1), applying an alternate projections scheme between the sets  $\mathcal{P}$  and  $\mathcal{Q}$ , similar to the one of Csiszár and Tusnády [3], but replacing the computation of the exact projections onto  $\mathcal{P}$  by suitable incomplete or approximate projections. In this paper we add to the IOP alternating algorithm the condition  $x^i \geq 0$ , then we define the sets:  $\mathcal{P}^+ = \mathcal{P} \cap \{x : x^i \geq 0\}$  and  $\mathcal{Q}^+ = \mathcal{Q} \cap \{x : x^i \geq 0\}$ , adopting the distance  $d(p, q) = \|p - q\|_D$ , for all  $p \in \mathcal{P}^+$ ,  $q \in \mathcal{Q}^+$ . We consider the following basic scheme:

**Algorithm 1** (*Basic Alternating Scheme*)

**Iterative step:** Given  $p^k = [x^k; r^k] \in \mathcal{P}^+$ ,  $x^k \geq 0$ , and  $q^k = [x^k; 0] \in \mathcal{Q}^+$ ,

find  $p_a^{k+1} = [x^{k+1}; r^{k+1}] \in \mathcal{P}^+$  as:

$$p_a^{k+1} \approx \arg \min \{ \|p - q^k\|_D^2 : p \in \mathcal{P}^+ \}, \quad \text{then}$$

define  $p^{k+1} = p_a^{k+1}$ , and  $q^{k+1} \in \mathcal{Q}^+$  by means of

$$q^{k+1} = [x^{k+1}; 0] \equiv \arg \min \{ \|p^{k+1} - q\|_D^2 : q \in \mathcal{Q}^+ \}.$$

In order to compute the incomplete projections onto  $\mathcal{P}^+$  we apply our DACCIM algorithm [4, 6]. Aiming at clarifying the applicability of DACCIM within the new approach, it is convenient to point out that, given a consistent system,  $\bar{A}y \leq \bar{b}$ , the sequence  $\{y^k\}$  generated by DACCIM, from the initial point  $y^0$ , converges to a solution  $\bar{y}^*$  of  $\bar{A}y \leq \bar{b}$ , satisfying

$$\bar{y}^* = \arg \min \{ \|y^* - y^0\|_D^2, \quad y^* \in \mathfrak{R}^n : \bar{A}y^* \leq \bar{b} \}.$$

This iterative algorithm uses simultaneous projections onto the hyperplanes defined by the rows of  $Ax - r = b$ , and the semi-spaces  $x^i \geq 0$ , if  $(x^i)^k < 0$ . In the following sections we will present the extended DIOP algorithm based on the same basic scheme of Algorithm 1, adding the conditions for accepting an approximate solution in  $\mathcal{P}^+$ . The theoretical properties of the new algorithm will be analyzed, and numerical experiences will be presented comparing its performance with some well-known methods.

## References

- [1] Y. Censor and S. Zenios, "Parallel Optimization: Theory and Applications," Oxford University Press, New York, 1997.
- [2] P. L. Combettes, *Inconsistent signal feasibility problems: least-squares solutions in a product space*, IEEE Trans. on Signal Processing, **42** (1994), 2955–2966.
- [3] I. Csiszár and G. Tusnády, *Information geometry and alternating minimization procedures*, Statistics and Decisions, Supplement, **1** (1984), 205–237.
- [4] Echebest, N., Guardarucci, M. T., Scolnik, H. D., Vacchino, M. C.: An acceleration scheme for solving convex feasibility problems using incomplete projection algorithms. Numerical Algorithms **35**, (2-4) 335-350 (2004).
- [5] H.D. Scolnik, N. Echebest, M.T. Guardarucci, M. C. Vacchino, *A class of optimized row projection methods for solving large non-symmetric linear systems*, Applied Numerical Mathematics, **41** (2002), 499–513.
- [6] H. D. Scolnik, N. Echebest, M. T. Guardarucci, M. C. Vacchino, *Incomplete Oblique Projections for Solving Large Inconsistent Linear Systems*, Mathematical Programming B, **111** (2008), 273–300.
- [7] Y. Xiao, D. Michalski, Y. Censor and J.M. Galvin, *Inherent smoothness of intensity patterns for intensity radiation therapy generated by simultaneous projection algorithms*, Physics in Medicine and Biology **49** (2004), 3227–3245.