A FEASIBLE DIRECTIONS METHOD FOR NONSMOOTH CONVEX OPTIMIZATION

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We consider the Problem

$$\min F(x),\ x \in \mathbb{R}^n,$$

where $F \in R$ is convex and not necessarily differentiable. Let be the Equivalent Problem EP:

$$\min z \in \mathbb{R}, \text{ such that } F(x) \leq z.$$

We present an algorithm that builds a sequence $\{(x_k, z_k)\}$ at the interior of the epigraph of F, such that $z_{k+1} < z_k$.

Our method employs the Feasible Directions Interior Point Algorithm, FD-IPA, for constrained smooth optimization, [1]. At each iteration, by solving two linear systems, FD-IPA computes a Feasible Descent Direction. Making a line search, a new interior point with a lower objective is obtained.

The present algorithm defines a sequence of Auxiliary Problems AP, where the constraints of EP are approximated by cutting planes. At each iteration a Search Direction for EP is obtained by computing with FD-IPA a Feasible Descent Direction of AP. If the step length is "short", AP is updated and a new search direction is computed. This procedure is repeated until a "good" step is obtained. When this happens, the search direction is a Feasible Descent Direction of EP.

The Algorithm

Parameters. $\xi,\ \mu \in (0,1),\ \varphi > 0,\ t_{max} > 0$.
Data. $(x^0, z^0) \in (\operatorname{epi} f)^\circ,\ \ f(x^0),\ \ y_0^0 = x^0,\ \ 0 < \lambda \in \mathbb{R}$,
$B \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ symmetric positive definite matrix, $k = 0$ and $l = 0$.
Step 1) Compute a subgradient $s_l^k \in \partial f(y_l^k)$. A new cutting plane at the current iterate (x^k, z^k) is defined by

$$g_l^k(x,z) = f(y_l^k) + (s_l^k)^T\ (x - y_l^k) - z, \qquad g_l^k(x,z) \in \mathbb{R}.$$

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Consider $\nabla g_l^k(x,z) = [(s_l^k)^T \quad -1]^T, \qquad \nabla g_l^k(x,z) \in R^{n+1}$.
Define

$$\bar{g}_l^k(x,z) = [g_0^k(x,z), ..., g_l^k(x,z)]^T, \qquad \bar{g}_l^k(x,z) \in \mathbb{R}^{l+1}$$

and $\nabla \bar{g}_l^k(x,z) = [\nabla g_0^k(x,z), ..., \nabla g_l^k(x,z)], \quad \nabla \bar{g}_l^k(x,z) \in \mathbb{R}^{(n+1)\times(l+1)}$.

Step 2) Calculation of a Feasible Descent Direction d^k for (AP_l)
i) Solve

$$B^k d_0 + \nabla \bar{g}_l^k(x^k, z^k)\lambda_0 = -\nabla z \tag{1}$$

$$\Lambda^k [\nabla \bar{g}_l^k(x^k, z^k)]^T d_0 + \bar{G}_l^k(x^k, z^k)\lambda_0 = 0. \tag{2}$$

If $d_0 = 0$, stop. else, solve

$$B^k d_1 + \nabla \bar{g}_l^k(x^k, z^k)\lambda_1 = 0 \tag{3}$$

$$\Lambda^k [\nabla \bar{g}_l^k(x^k, z^k)]^T d_1 + \bar{G}_l^k(x^k, z^k)\lambda_1 = -\lambda, \tag{4}$$

where $\bar{G}_l^k(x,z) \equiv \operatorname{diag}[g_1^k(x,z), ..., g_l^k(x,z)]$.
ii) If $d_1^T \nabla z > 0$, set $\rho = \varphi \|d_0\|^2$.
Otherwise, set $\rho = \min\left\{\varphi\|d_0\|^2, (\xi - 1)\frac{d_0^T \nabla z}{d_1^T \nabla z}\right\}$.
iii) Compute the feasible descent direction

$$d_l^k = d_0 + \rho d_1 \quad \text{and} \quad \bar{\lambda}^k = \lambda_0 + \rho \lambda_1.$$

Step 3) Compute

$$t_l^k = \min\left\{t_{max}, \ \max\{t \mid \bar{g}_l^k((x^k, z^k) + t d_l^k) \le 0\}\right\}.$$

Step 4) Set

$$(y_{l+1}^k, w_{l+1}^k) = (x^k, z^k) + \mu t_l^k d_l^k.$$

If $w_{l+1}^k \le f(y_{l+1}^k)$, the step is not serious. Then, set $l := l + 1$, define $0 < \lambda_k^{l+1}$ and go to Step 1.

Else, the step is serious. Then, define $(x^{k+1}, z^{k+1}) = (y_{l+1}^k, w_{l+1}^k)$, $0 > \lambda_{k+1}^0$, set $k = k + 1$, $l = 0$, $y_0^k = x^k$ and go to Step 1. □

We prove global convergence of the present algorithm. Several test problems are solved very efficiently.

[1] Herskovits, J. *Feasible direction interior-point technique for nonlinear optimization. Journal of Optimization Theory and Applications 99*, 1 (1998), 121–146.