Factorization of overdetermined boundary value problems

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Abstract

In this talk we present the application of the factorization method of linear elliptic boundary value problems to overdetermined problems. Let Ω be the cylinder $\Omega =]0,1[\times\mathcal{O}$, $x' = (x,y) \in \mathbb{R}^n$, where x is the coordinate along the axis of the cylinder and $\mathcal{O}$, a bounded open set in $\mathbb{R}^{n-1}$, is the section of the cylinder. Let $\Sigma =]0,1[\times\partial\mathcal{O}$ be the lateral boundary, $\Gamma_0 = \{0\}\times\mathcal{O}$ and $\Gamma_1 = \{1\}\times\mathcal{O}$ be the faces of the cylinder.

We consider the following Poisson equation with mixed boundary conditions

$$(\mathcal{P}_0)\begin{cases} -\Delta z = -\dfrac{\partial^2 z}{\partial x^2} - \Delta_y z = f \quad \text{in } \Omega, \\ z|_\Sigma = 0, \\ -\dfrac{\partial z}{\partial x}|_{\Gamma_0} = z_0, \quad z|_{\Gamma_1} = z_1. \end{cases} \tag{1}$$

If $f \in L^2(\Omega)$, $z_0 \in (H^{1/2}_{00}(\mathcal{O}))'$ and $z_1 \in H^{1/2}_{00}(\mathcal{O})$, problem $(\mathcal{P}_0)$ has an unique solution in

$$\mathcal{Z} = \{z \in H^1(\Omega) : \Delta z \in L^2(\Omega), z|_\Sigma = 0\}.$$

We can write problem $\mathcal{P}_0$ in matrix form as follows:

$$\mathcal{A}\begin{pmatrix} p \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad z(1) = z_1, \ p(0) = -z_0, \tag{2}$$

with

$$\mathcal{A} = \begin{pmatrix} -I & \frac{\partial}{\partial x} \\ -\frac{\partial}{\partial x} & -\Delta_y \end{pmatrix}.$$

Assuming we have an extra information, given by a Neumann boundary condition at point 1, we consider the overdetermined system

$$\mathcal{A}\begin{pmatrix} p \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}, \ z(1) = z_1, \ p(0) = -z_0, \ \frac{\partial z}{\partial x}(1) = z_2, \tag{3}$$

where $z_0 \in H^{1/2}_{00}(\mathcal{O})$, $z_1 \in H^{3/2}_0(\mathcal{O})$, $z_2 \in H^{1/2}_{00}(\mathcal{O})$, $f \in H^{5/2}(\Omega)$ and $(\Delta f)|_\Sigma = 0$. If the data are not compatible with (2), this system should be satisfied in the least square sense.

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We introduce a perturbation and, by minimizing the norm of the perturbation subject to the constraint given by (3), we obtain the normal equation for the overdetermined problem as

$$(\mathcal{P}_2)\quad \begin{cases} \Delta^2 z = -\Delta f, \quad \text{in } \Omega, \\ z|_\Sigma = 0, \quad \Delta z|_\Sigma = 0, \\ -\dfrac{\partial z}{\partial x}(0) = z_0, \quad \dfrac{\partial \Delta z}{\partial x}(0) = -\dfrac{\partial f}{\partial x}(0), \\ z(1) = z_1, \ \dfrac{\partial z}{\partial x}(1) = z_2. \end{cases} \tag{4}$$

We factorize problem $(\mathcal{P}_2)$ by invariant embedding and obtain the decoupled system

$$\frac{dP}{dx} + P^2 + \Delta_y = 0, \quad P(0) = 0, \tag{5}$$

$$\frac{dQ}{dx} + PQ + QP = I, \quad Q(0) = 0, \tag{6}$$

$$\frac{\partial t}{\partial x} + Pt = -\Delta f, \quad t(0) = -\frac{\partial f}{\partial x}(0), \tag{7}$$

$$\frac{\partial \tilde{r}}{\partial x} + P\tilde{r} = -Qt, \quad \tilde{r}(0) = -z_0, \tag{8}$$

$$\frac{\partial \Delta z}{\partial x} - P\Delta z = t, \quad \Delta z(1) = c, \tag{9}$$

$$\frac{\partial z}{\partial x} - Pz = Q\Delta z + \tilde{r}, \quad z(1) = z_1. \tag{10}$$

P is the Dirichlet to Neumann operator. For each $s \in [0,1]$, $Q(s)$ is an operator from $(H_{00}^{1/2}(\mathcal{O}))'$ into $H_{00}^{1/2}(\mathcal{O})$ and from $L^2(\mathcal{O})$ into $H_0^1(\mathcal{O})$. Moreover, $Q(s)$ is a linear, self-adjoint and positive operator in $L^2(\mathcal{O})$. It is easy to see, from the definition, that $Q(1)$ is a bijective operator from $(H_{00}^{\frac{1}{2}}(\mathcal{O}))'$ to $H^{\frac{1}{2}}(\mathcal{O})$, so we can define $(Q(1))^{-1}$. We finally evaluate

$$c = (Q(1))^{-1}(z_2 - P(1)\ z_1 - \tilde{r}(1)). \tag{11}$$

Once again we can remark the interest of the factorized form if the same problem has to be solved many times for various sets of data (z_1, z_2). Once the problem has been factorized, that is, P and Q have been computed, and t and $\tilde{r}$ are known, the solution for a data set (z_1, z_2) is obtained by solving (11) and then the Cauchy initial value problems (9), (10) backwards in x.