Asymptotic expansion of Dirichlet to Neumann operators on slim bodies

J. Henry[*] B. Louro[†] A. M. Ramos[‡]

Abstract

The factorization method for boundary value problems furnishes the Dirichlet to Neumann operator on a part of the boundary. In this talk slim bodies are considered, more precisely, we consider cylinders $\Omega^\varepsilon =]0,a[\times\mathcal{O}^\varepsilon$ in $\mathbb{R}^d$, with $\mathcal{O}^\varepsilon = \varepsilon\mathcal{O}$ and $\mathcal{O}$ a bounded open set in $\mathbb{R}^{d-1}$ ($d = 2$ or 3 in real applications). Parameter ε denotes that the section $\mathcal{O}^\varepsilon$ is much smaller than the length of the axis a. We denote $\Gamma_s^\varepsilon = \{s\}\times\mathcal{O}^\varepsilon$, the lateral boundary of the cylinder $\Sigma^\varepsilon =]0,a[\times\partial\mathcal{O}^\varepsilon$ and a general point $(x_1^\varepsilon, x_2^\varepsilon, \cdots, x_d^\varepsilon) \in \Omega^\varepsilon$ is also denoted by $(x^\varepsilon, y^\varepsilon)$, where $x^\varepsilon = x_1^\varepsilon$ and y^ε denotes the independent variables $(x_2^\varepsilon, ..., x_d^\varepsilon)$. Let $f \in L^2(\Omega^\varepsilon)$, $u_0 \in H^{1/2}(\mathcal{O}^\varepsilon)$, $u_a \in H^{1/2}(\mathcal{O}^\varepsilon)'$ and λ be a positive constant. The problem we want to solve is

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } \Omega^\varepsilon, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Sigma^\varepsilon, \\ u = u_0 & \text{on } \Gamma_0^\varepsilon, \\ \dfrac{\partial u}{\partial x} = u_a & \text{on } \Gamma_a^\varepsilon, \end{cases} \tag{1}$$

where ν represents the outward normal vector on the boundary.

We want to study the asymptotic behavior of the solution when $\varepsilon \to 0$. We do first a change of variable so that the new domain is the same for all $\varepsilon > 0$. We consider the domain given by the cylinder $\Omega =]0,a[\times\mathcal{O}$ and the change of variable $(x^\varepsilon, y^\varepsilon) = (x, \varepsilon y)$. Therefore, in the new variables (x, y), (1) can be re-written equivalently by

$$\begin{cases} -\dfrac{\partial^2 u}{\partial x^2} - \dfrac{1}{\varepsilon^2}\Delta_y u + \lambda u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Sigma, \\ u = u_0 & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial x} = u_a & \text{on } \Gamma_a, \end{cases} \tag{2}$$

Factorizing problem (2) by invariant embedding techniques, we arrive to the uncoupled system:

$$\begin{cases} -\dfrac{dQ}{dx} + Q^2 = \lambda I + \dfrac{1}{\varepsilon^2}\mathcal{A}, \\ Q(a) = 0 \end{cases} \tag{3}$$

$$\begin{cases} -\dfrac{dw}{dx} + Qw = -f, \\ w(a) = -u_a \end{cases} \tag{4}$$

[*]INRIA Bordeaux Sud Ouest, France (jacques.henry@inria.fr)
[†]FCT and CMA, Universidade Nova de Lisboa, Portugal (bjl@fct.unl.pt)
[‡]Departamento de Matemática Aplicada, Universidad Complutense de Madrid, España (angel@mat.ucm.es)

$$\begin{cases} \dfrac{du}{dx} + Qu = -w, \\ u(0) = u_0, \end{cases} \tag{5}$$

where $(\mathcal{A}h, \varphi) = (\nabla_y h, \nabla_y \varphi) \quad \forall \; h, \varphi \in H^1(\mathcal{O})$. $\mathcal{A}$ is the abstract operator corresponding to Neumann boundary conditions for the laplacian. Q is the Dirichlet to Neumann operator.

We present an asymptotic expansion of this factorized version of the problem and equations satisfied by each term of the expansion.