

# A POSTERIORI ERROR ESTIMATION AND ADAPTIVITY FOR STATE CONSTRAINED PARABOLIC OPTIMAL CONTROL PROBLEMS

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In this paper we consider state constrained optimal control problems (OCPs) governed by a parabolic state equation. The focus lies on the efficiency of the numerical solution, achieved by a local refinement strategy, which in turn is guided by an a posteriori estimation of the error in the cost functional.

When solving complex optimal control problems, finding an effective discretization of the involved quantities is desirable, and the use of adaptive techniques based on a posteriori error estimation is well established. For this, it is possible to consider the error in the natural norms of the corresponding spaces, like in [4], or alternatively the error in the cost functional. For an OCP with cost functional  $J$  and an elliptic state equation coupling the control  $q$  and the state  $u$ , considering an FE discretization of the problem with optimal solution  $(q_h, u_h)$ , this would mean estimating  $J(q, u) - J(q_h, u_h)$ , or cellwise contributions thereof. In [1], the authors present a general concept for a posteriori estimation of the discretization error with respect to the cost functional in the context of optimal control problems.

While this publication dealt with unconstrained problems, e.g., in [10] control constraints are considered in the case of elliptic state equations. Constraints on the state variable, on the other hand, require different treatment due to the reduced regularity of the optimal solution, and were considered in [2, 3, 11].

Treating parabolic OCPs, additional aspects arise. Firstly, a discretization in space and time is to be conducted, e.g., like in [6, 7], where unconstrained problems are considered. We will even use *dynamic* meshes, as described in [9], where in every time step a different spatial mesh may be utilized. Clearly this requires a partition of the estimator in the corresponding parts, the spatial and temporal discretization errors. Regularization of state constrained parabolic OCPs has been addressed, e.g., in [8]. A posteriori error estimation and the following adaptive refinement strategy for this problem class in turn has to our knowledge not been studied before. So in this paper we will consider the model problem

$$\min J(q, u)$$

subject to  $q \in Q, u \in X_{ad}$ , with

$$X_{ad} = \{u \in W(0, T) \cap C([0, T] \times \bar{\Omega}) : \\ u_a \leq u(t, x) \leq u_b\},$$

and

$$u = S(q)$$

where  $S$  is the solution operator of a parabolic partial differential equation,  $W(0, T) = \{v \in L^2((0, T), V), \partial_t v \in L^2((0, T), V^*)\}$  and  $Q, V$  are suitably chosen spaces.

Applying an interior-point-method, and a finite element discretization to the resulting regularized problem will lead to a discrete solution  $(q_\sigma, u_\sigma)$ . We will derive an error estimator

$$J(q, u) - J(q_\sigma, u_\sigma) \approx \eta_\varepsilon + \eta_k + \eta_h + \eta_d$$

consisting of the regularization, and the temporal, spatial, and control space discretization errors, and advise an equilibration strategy for these error contributions.

To demonstrate the behavior of this adaptive algorithm we will consider a numerical example obtained from the practical problem of crack prevention in young concrete: In the hydration phase of young concrete, chemical processes release heat inside the concrete structure, leading to internal tensile stresses. In some constellations, this might cause cracks in the structure. The goal of the optimization is to adjust the course of action (concrete composition, water cooling, initial and boundary conditions) in the least costly way such that no cracks will occur. For comprehensive review on young concrete see, e.g., [5].

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