Well-posedness and regularity of the the problem of transmission of the Schrödinger equation

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Let Ω and Ω_1 be two bounded domain of $\mathbb{R}^n$ $(n \geq 2)$ with boundary Γ and Γ_1 of class C^2 such that

$$\overline{\Omega_1} \subset \Omega$$

We assume that Ω_1 is simply connected and we set

$$\Omega_2 = \Omega \backslash \Omega_1.$$

We have therefore

$$\partial \Omega_2 = \Gamma \cup \Gamma_1$$

Let a time $T > 0$ and two distinct constants $a_1, a_2 > 0$ be given.

In this paper, we shall be concerned with the following system of transmission of the Schrödinger equation with Dirichlet control and colocated observation.

$$y_k'(x,t) = \mathbf{i} a_k \Delta y_k(x,t), \quad k = 1,2, \qquad (x,t) \in \Omega_k \times (0,T) \tag{1.1}$$

$$y_k(x,0) = y_k^0(x), \qquad k = 1,2 \quad x \in \Omega_k \tag{1.2}$$

$$y_2(x,t) = u(x,t), \qquad (x,t) \in \Gamma \times (0,T) \tag{1.3}$$

$$y_1(x,t) = y_2(x,t), \qquad (x,t) \in \Gamma_1 \times (0,T) \tag{1.4}$$

$$a_1 \frac{\partial y_1(x,t)}{\partial \nu} = a_2 \frac{\partial y_2(x,t)}{\partial \nu}, \qquad (x,t) \in \Gamma_1 \times (0,T) \tag{1.5}$$

$$z(x,t) = \mathbf{i} \frac{\partial}{\partial \nu}(A^{-1} y_2(x,t)), \qquad (x,t) \in \Gamma \times (0,T) \tag{1.6}$$

where
- $y'_k(x,t) = \frac{\partial y_k(x,t)}{\partial t}$
- ν is the unit normal on Γ_1 pointing towards the exterior of Ω_2.
- $A : H^{-1}(\Omega) \rightarrow H^{-1}(\Omega)$ is a positive selfadjoint operator defined by

$$Af = \Delta f, \qquad D(A) = H_0^1(\Omega)$$

- u is the input function.
- z is the output function.

The aim of this paper is to show that the system (1.1)-(1.6) belongs to the class of regular well-posed linear systems introduced by Salamon and Weiss ([1], [2], [3]). The main results are

Theorem 1 *The equations (1.1)-(1.6) determine a well-posed linear system with input and ouput space $U = L^2(\Gamma)$ and state space $X = H^{-1}(\Omega)$.*

Theorem 2 *The system (1.1)-(1.6) is regular with zero feedthrough operator. This means that if the intial state $y(.,0) = 0$ and $u(.,t) = u(t) \in U$ is a step input then the corresponding output satisfies*

$$\lim_{\sigma \to 0} \int_\Gamma \left| \frac{1}{\sigma} \int_0^\sigma z(x,t)dt \right| d\Gamma = 0$$

References

[1] D. Salamon, Infinite-dimensional systems with unbounded control and observation: a functional analytic approach, *Trans. Amer. Math. Soc.* 300 (1987), 383-431.

[2] G. Weiss, Regular linear systems with feedback, *Math. Control Signals Systems* 7 (1994), 23-57.

[3] G. Weiss, Transfer functions of regular linear systems I: characterizations of regularity, *Trans. Amer. Math. Soc.* 342 (1994), 827-854.