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NUMERICAL ANALYSIS OF DISTRIBUTED OPTIMAL CONTROL ON THE INTERNAL ENERGY IN MIXED ELLIPTIC PROBLEMS

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ABSTRACT

We consider a bounded domain Ω in $\mathbb{R}^n$, whose regular boundary Γ consists of the union of two disjoint portions Γ_1 y Γ_2 with meas$(\Gamma_1) > 0$ and meas$(\Gamma_2) > 0$. We denote with meas (Γ) the (n-1)-dimensional Lebesgue measure of Γ.

We consider the following two steady-state heat conduction problems P and P$_\alpha$ (for each parameter $\alpha > 0$) respectively with mixed boundary conditions:

(1) $$-\Delta u = g \;\text{ in }\Omega \qquad u\big|_{\Gamma_1} = b \qquad -\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q$$

and

(2) $$-\Delta u = g \;\text{ in }\Omega \qquad -\frac{\partial u}{\partial n}\Big|_{\Gamma_1} = \alpha\,(u-b) \qquad -\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q$$

where g is the internal energy in Ω, b is the temperature on Γ_1 for (1) and the temperature of the external neighboord of Γ_1 for (2), q is the heat flux on Γ_2 and $\alpha > 0$ is the heat transfer coefficient of Γ_1 (Newton´s law on Γ_1), that satisfy the following assumptions:

(3) $$g \in H = L^2(\Omega)\,, \qquad q \in L^2(\Gamma_2)\,, \qquad b \in H^{\frac{1}{2}}(\Gamma_1).$$

Problems (1) and (2) can be conisdered as the steady-state Stefan problem for suitable data q, g and b. The variational inequalities corresponding to the problems (1) and (2) are given respectively by

(3) $$u \in K \;:\; a(u,\,v) = (g,\,v)_H - \int_{\Gamma_2} q\,v\,ds,\; \forall\, v \in V_0,$$

(4) $$u \in V \;:\; a_\alpha(u,\,v) = (g,\,v)_H - \int_{\Gamma_2} q\,v\,ds \,+\, \alpha\int_{\Gamma_1} b\,v\,ds\,,\; \forall\, v \in V,$$

where :

(5) $$V = H^1(\Omega),\; H = L^2\,(\Omega), \quad V_0 = \left(v \in V \big/ v\big|_{\Gamma_1} = 0\right), \quad K = V_0 + v_0\,(v_0 \in V/v\big|_{\Gamma_1} = b),$$

(6) $$a(u,v) = \int_\Omega \nabla u \nabla v\,dx, \qquad a_\alpha(u,v) = \int_\Omega \nabla u \nabla v\,dx + \alpha\int_{\Gamma_1} uv\,ds, \qquad (u,v)_H = \int_\Omega uv\,dx\,.$$

We consider g as a control variable for the cost functional $J : H \to \mathbb{R}_0^+$ and $J_\alpha : H \to \mathbb{R}_0^+$ respectively given by:

(7) $$J(g) = \tfrac{1}{2}\|u_g - z_d\|_H^2 + \tfrac{M}{2}\|g\|_H^2 \quad\text{and}\quad J_\alpha(g) = \tfrac{1}{2}\|u_{g\alpha} - z_d\|_H^2 + \tfrac{M}{2}\|g\|_H^2$$

where u_g and $u_{g\alpha}$ are the unique solutions of the variational inequalities (3) and (4) respectively, $z_d \in H$ is given and $M = C$onst. > 0. We formulate the following continuous distributed optimal control problems:

(8) Find $g_{op} \in H$ such that $J(g_{op}) = \underset{g \in H}{Min} \; J(g)$,

and

(9) Find $g_{op_\alpha} \in H$ such that $J_\alpha(g_{op_\alpha}) = \underset{g \in H}{Min} \; J_\alpha(g)$.

In C.M. Gariboldi - D.A. Tarzia, Applied Mathematics and Optimization, 47 (2003), 213-230 we prove that the functionals J and J_α are coercive (then strictly convex) and Gâteaux differentiable on H, and that J' and J'_α are lipschitzian and strictly monotone applications on H. We also prove the existence and uniqueness of the distributed optimal control problems (8) and (9) and we characterize this optimal energy g_{op} and g_{op_α} as a fixed point on H of suitable operators W and W_α over his adjoint state p_g and $p_{g\alpha}$ respectively for large parametrer M. Moreover, we study the convergence when $\alpha \to \infty$ of the optimal control problem (9) corresponding to the state system (4).

The goal of this paper is to study the numerical analysis of the optimal control problems (8) and (9) and its corresponding convergence by using the finite element theory. We consider b=Const. > 0 and τ_h a regular triangulation of the polygonal domain Ω with Lagrange triangles of type 1, constituted by affine-equivalent finite element of class C^0, where the parameter $h > 0$ of the finite element approximation goes to zero. We can take h equal to the longest side of the triangles $T \in \tau_h$ and we can approximate V_0 by:

$$(10) \qquad V_{0h} = \left(v_h \in C^0(\Omega) / v_h \big|_T \in P_1(T) \; \forall T \in \tau_h, \; v_h \big|_{\Gamma_1} = 0 \right)$$

where P_1 is the set of the polynomials of degree less than or equal to 1. We consider g as a control variable for the discreted cost functionals $J_h : H \to \mathbb{R}_0^+$ and $J_{h\alpha} : H \to \mathbb{R}_0^+$ respectively given by:

$$(11) \qquad J_h(g) = \tfrac{1}{2}\|u_{hg} - z_d\|_H^2 + \tfrac{M}{2}\|g\|_H^2 \quad \text{and} \quad J_{h\alpha}(g) = \tfrac{1}{2}\|u_{hg\alpha} - z_d\|_H^2 + \tfrac{M}{2}\|g\|_H^2$$

where u_{hg} and $u_{hg\alpha}$ are the unique solutions of the discretized variational inequalities (3) and (4) respectively. Then, we can formulate the following discrete distributed optimal control problems:

(12) Find $g_{hop} \in H$ such that $J_h(g_{hop}) = \underset{g \in H}{Min} \; J_h(g)$,

and

(13) Find $g_{hop_\alpha} \in H$ such that $J_{h\alpha}(g_{hop_\alpha}) = \underset{g \in H}{Min} \; J_{h\alpha}(g)$.

We prove that the functional J_h and $J_{h\alpha}$ are coercive, strictly convex and Gâteaux differentiable on H, and that J'_h and $J'_{h\alpha}$ are lipschitzian and strictly monotone applications on H. We also prove the existence and uniqueness of the distributed optimal control problems (12) and (13) and we characterize this optimal energy g_{hop} and g_{hop_α} as a fixed point on H of suitable operators W_h and $W_{h\alpha}$ over his adjoint state p_{hg} and $p_{hg\alpha}$ respectively for large parametrer M. Moreover, we obtain the folowing results between the continuous and discrete distributed optimal control problems (8) and (12) respectively for all positive parameter M, given by:

$$(14) \qquad \lim_{h \to 0} \|u_{hgop} - u_{gop}\|_V = 0, \; \lim_{h \to 0} \|p_{hgop} - p_{gop}\|_V = 0, \; \lim_{h \to 0} \|g_{hop} - g_{op}\|_H = 0,$$

where u_{hgop} and p_{hgop} are the system and the adjoint states of the discreted optimal control problem (12) respectively. A similar results can be obtained for the continuous and discrete distributed optimal control problems (9) and (13) respectively for all positive parameter M.