Solution of a Singular Control Problem with Delays

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The singular control problem is an optimal control problem, for which the Pontryagin's Maximum Principle is not applicable. This means that the Pontryagin's Maximum Principle fails to yield candidate optimal controls for this problem. In such a case, either higher order necessary or sufficient optimality conditions can be helpful in solving the problem (see e.g. [3, 4, 8, 9] and references therein). However, such conditions fail to yield a candidate optimal control for the problem, having no solution (an optimal control) in the class of regular functions, even if the cost functional has either a finite infinum (the case of minimizing the cost functional) or supremum (the case of maximizing the cost functional) in this class of functions. Such problems with undelayed dynamics were studied extensively in the literature. Two main approaches have been proposed to solve these problems. In the first approach, the optimal control is sought in a properly defined class of generalized functions [11]. In the second approach, a minimizing (maximizing) sequence of controls is derived, i.e., a sequence of regular control functions, along which the cost functional tends to its infinum (supremum) (see e.g. [7, 9] and references therein). One of the methods of constructing such a sequence is the regularization method [3]. Singular control problems with delayed dynamics, which have no solutions in the class of regular functions, have been considered much less in the literature [1, 2, 10]. In this works, a linear-quadratic optimal control problem with a single point-wise state delay in the dynamics was considered. The case, where the performance index does not contain a control cost, was studied. The optimal control was sought in some class of generalized functions.

In this presentation, a singular control problem with state delays in the dynamics is studied. A minimizing sequence for this problem is constructed by using the regularization approach and singular perturbation technique.

The following linear-quadratic optimal control problem with point-wise and distributed state delays in the dynamics is considered

$$dx(t)/dt = A_1(t)x(t) + A_2(t)y(t) + H_1(t)x(t-h) + H_2(t)y(t-h) + \int_{-h}^{0}\Big[G_1(t,\tau)x(t+\tau) + G_2(t,\tau)y(t+\tau)\Big]d\tau, \quad (1)$$

$$dy(t)/dt = A_3(t)x(t) + A_4(t)y(t) + H_3(t)x(t-h) + H_4(t)y(t-h) + \int_{-h}^{0}\Big[G_3(t,\tau)x(t+\tau) + G_4(t,\tau)y(t+\tau)\Big]d\tau + u(t), \quad (2)$$

$$x(\tau) = \varphi_x(\tau),\ \ y(\tau) = \varphi_y(\tau),\ \tau \in [-h, 0);\quad x(0) = \varphi_{x0},\ \ y(0) = \varphi_{y0}, \quad (3)$$

$$J(u(t)) \triangleq \int_0^T \Big[x'(t)D_x(t)x(t) + y'(t)D_y(t)y(t)\Big] \to \min_{u(t)\in U_1}, \quad (4)$$

where $x(t) \in E^n$, $y(t) \in E^m$, $u(t) \in E^m$, ($u(t)$ is a control function); $A_i(t)$, $H_i(t)$, $G_i(t,\tau)$, $(i = 1, ..., 4)$ and $\varphi_x(\tau)$, $\varphi_y(\tau)$, φ_{x0}, φ_{y0} are given matrices and vectors of corresponding dimensions; $h > 0$ is a given constant time delay; $T > 0$ is a prescribed time-instant; $D_x(t)$ and $D_y(t)$ are given symmetric positive semi-definite and positive definite matrices of corresponding dimensions; the prime (as a superscript) denotes the transposition; $U_1 = L^1[0, T; E^m]$.

Since the cost functional in (4) does not contain a quadratic control cost, the problem (1)-(4) is a singular control problem. This problem has no, in general, an optimal control among regular functions. However, due to the assumptions on the matrices $D_x(t)$ and $D_y(t)$, the cost functional in this problem has a finite infinum

$$J^* = \inf_{u(t)\in U_1} J(u(t)). \tag{5}$$

Due to the regularization method, the original problem (1)-(4) is associated with a new linear-quadratic cheap control problem, i.e. the optimal control problem consisting of the equations of dynamics (1)-(2), the initial conditions (3) and the following performance index

$$J_\varepsilon(u(t)) \triangleq \int_0^T \left[x^{'}(t)D_x(t)x(t) + y^{'}(t)D_y(t)y(t) + \varepsilon^2 u^{'}(t)u(t)\right] \to \min_{u(t)\in U_2\subset U_1}, \tag{6}$$

where $\varepsilon > 0$ is a small parameter; $U_2 = L^2[0,T;E^m]$.

Cheap control problems with undelayed dynamics have been studied extensively in the literature, while the delayed dynamics problems were investigated much less (see e.g. [5, 6] and references therein). Under some smoothness assumptions on the data of the original problem (1)-(4), by using the singular perturbation technique, an asymptotically suboptimal control $u_{so}(t,\varepsilon)$ is constructed for the cheap control problem (1)-(2),(3),(6). Then, it is shown that this control is minimizing for the original problem (1)-(4), i.e.,

$$\lim_{\varepsilon\to+0} J(u_{so}(t,\varepsilon)) = J^*. \tag{7}$$

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