ASYMPTOTIC EXPANSION OF THE SOLUTION OF A CONTROL CONSTRAINED LINEAR QUADRATIC OPTIMAL CONTROL PROBLEM WITH INTERIOR PENALTY

FELIPE ÁLVAREZ*, J. FRÉDÉRIC BONNANS†, JÉRÔME BOLTE‡, AND FRANCISCO J. SILVA†

Interior-point methods for solving constrained finite-dimensional optimization problems based on the logarithmic penalty are recognized as being presently among the most efficient algorithms (see [5]).

These methods are also specially well-suited for constrained optimal control problems since the resulting problem, when variables are ordered over time, have a Jacobian with a band structure, for which there exists e.g. QR factorization software. The corresponding approach has been applied to real-world aerospace optimization problems (see [2]). In addition, it can be proved that for unconstrained problems (to which interior-point methods reduce the original ones) discretization can be analyzed and evaluated with a good precision, allowing to design efficient grid refinement algorithms (see [1, 3, 4]).

Yet a missing step is the analysis of the error performed when using interior penalty. There are some results on this topic, including the linear convergence of a short-step path-following algorithm and $O(\sqrt{\varepsilon})$ error estimates (see [6]).

In this work we will focus on a question formulated in [1] concerning the error of the solution of a penalized problem for linear-quadratic optimal control problems. Specifically, for the model example

$$\begin{cases} \text{Min}\ \frac{1}{2}\displaystyle\int_0^1 (u(t)-2t)^2 \mathrm{d}t + \frac{1}{2}y(1)^2, \\ \dot{y}(t) = u(t); \quad u(t) \geq 0,\ t \in [0,1]; \quad y(0) = \frac{3}{4}. \end{cases} \tag{0.1}$$

the system of equations obtained as the first order conditions for the penalized problem

$$\begin{cases} \text{Min}\ \frac{1}{2}\displaystyle\int_0^1 (u(t)-2t)^2 - \varepsilon \log u(t)\ \mathrm{d}t + \frac{1}{2}y(1)^2, \\ \dot{y}(t) = u(t); \quad y(0) = \frac{3}{4}. \end{cases} \tag{0.2}$$

is not differentiable with respect to the penalization parameter at $\varepsilon = 0$. This is an important difficulty since it is not possible to apply the standard Implicit Function Theorem. Instead, they obtain, after an application of the so-called Restoration theorem, an asymptotic expansion for the adjoint state p_ε of the type

$$p_\varepsilon(t) = p_0(t) + c(t)\varepsilon\log\varepsilon + O((\varepsilon\log\varepsilon)^2). \tag{0.3}$$

where $c : [0,T] \to \mathbb{R}$ is a continuous function. This in turn implies that

$$||u_\varepsilon - u_0||_{L^\infty} = O(\sqrt{\varepsilon}) \tag{0.4}$$

$$||u_\varepsilon - u_0||_{L^1} = O(\varepsilon\log\varepsilon) \tag{0.5}$$

---

[1] Department of Mathematical Engineering, Universidad de Chile & Centre for Mathematical Modelling, Universidad de Chile-CNRS. Casilla 170-3 Santiago 3, Chile.

[2] INRIA-Saclay & CMAP, École Polytechnique, 91128 Palaiseau, France.

[3] Université Paris 06, Equipe Combinatoire et Optimisation, UMR 7090, Case 189, 75252 Paris, France and INRIA-Saclay & CMAP, École Polytechnique, 91128 Palaiseau, France.

We generalize this analysis to general linear quadratic optimal control problems and propose a method for evaluating the error that works in a very general framework. Finally, we obtain a “first-order” expansion of the path, allowing to understand how the variations of the optimal control are related to the junction points (times where the set of active constraints changes). In addition our results are of a general nature, and deal with a general interior penalty.

REFERENCES

[1] N. Bérend, J.F. Bonnans, J. Laurent-Varin, M. Haddou, and C. Talbot. Fast linear algebra for multiarc trajectory optimization. In G. Di Pillo and M. Roma, editors, *Large scale nonlinear optimization*, volume 83 of *Nonconvex Optimization and Its Applications*, pages 1–14. Springer, 2006.

[2] N. Bérend, J.F. Bonnans, J. Laurent-Varin, M. Haddou, and C. Talbot. An interior-point approach to trajectory optimization. *AIAA J. of Guidance, Control and Dynamics*, 30(5):1228–1238, 2007.

[3] M. Bergounioux, M. Haddou, M. Hintermüller, and K. Kunisch. A comparison of a Moreau-Yosida-based active set strategy and interior point methods for constrained optimal control problems. *SIAM Journal on Optimization*, 11:495–521 (electronic), 2000.

[4] J.F. Bonnans and J. Laurent-Varin. Computation of order conditions for symplectic partitioned runge-kutta schemes with application to optimal control. *Numerische Mathematik*, 103(1):1–10, 2006.

[5] Clovis C. Gonzaga. Path-following methods for linear programming. *SIAM Rev.*, 34(2):167–224, 1992.

[6] M. Weiser. Interior point methods in function space. *SIAM J. on Control and Optimization*, 44:1766–1786 (electronic), 2005.