Linear-Quadratic Differential Games Revisited

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Abstract

The object of this paper is to revisit the results of Bernhard (*J. Optim. Theory Appl. 27 (1979), 51–69*) on two-person zero-sum linear quadratic differential games and generalize them to utility functions without positivity assumptions on the matrices acting on the state variable and to linear dynamics with bounded measurable data matrices. The paper specializes to *state feedback* via Lebesgue measurable *affine closed loop strategies* with possible non L^2-integrable singularities. After sharpening our recent results of Delfour (*SIAM J. Control Optim. 46 (2007), 750–774*) on the characterization of the open loop lower and upper values of the game, it first deals with L^2-integrable closed loop strategies and then with the larger family of strategies that may have non L^2-integrable singularities. A new conceptually meaningful and mathematically precise definition of a closed loop saddle point is introduced to simultaneously handle state feedbacks of the L^2-type and smooth locally bounded ones except at most in the neighborhood of finitely many instant of time. A necessary and sufficient condition is that the free end problem be *normalizable almost everywhere*. This relaxation of the classical notion of normalizability allows singularities in the feedback law at an infinite number of instants including accumulation points that are not isolated. A complete classification of closed loop saddle points is given in terms of the convexity/concavity properties of the utility function and connections are given with the open loop lower value, upper value, and value of the game.

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1 Extended abstract

The object of this paper is to revisit the pioneering work of P. Bernhard [2] on two-person zero-sum linear quadratic differential games and generalize it to utility functions without positivity assumptions on the matrices acting on the state variable and to linear dynamics with bounded measurable data matrices. The paper specializes to *state feedback* via Lebesgue measurable *affine closed loop strategies* with possible non L^2-integrable singularities. After sharpening the recent results of M. C. Delfour [3] on the characterization of the open loop lower and upper values of the game, it first deals with L^2-integrable closed loop strategies and then with the larger family of strategies that may have non L^2-integrable singularities.

Several equivalent necessary and sufficient conditions are given for the existence of a closed loop saddle point with respect to L^2-integrable affine closed loop strategies: for instance, the *normality* of the problem; the existence of an $H^1(0,T)$ solution to the associated matrix Riccati differential equation. It was shown in [3] that the existence of a solution to the coupled state-adjoint state system is a necessary condition for the existence of a finite open loop lower value, upper value, or value of the game and that the difference essentially depends on the convexity of the utility function with respect to the control of the minimizing player and on its concavity with respect to the control of the maximizing player. This condition is also necessary for the existence of a closed loop saddle point. It leads to a complete classification in terms on the convexity/concavity properties of the utility function.

The paper deals with two delicate issues. The first one is the very definition of a closed loop saddle point in the presence of closed loop strategies with non L^2-integrable singularities. As was pointed out in [2, p. 68 and Remark 5.1] such strategies may lead to conflicting terms that simultaneously blow up in the utility function. Under the positivity assumptions, one may possibly get around this problem by setting the utility function equal to $\pm\infty$, but we don't have them here. So we had to introduce a *new conceptually meaningful and mathematically precise* definition. It says that the original problem can be transformed via feedback in such a way that the resulting problem has an open loop saddle point at $(0,0)$. The second related issue was to specify the class of affine closed loop strategies in such a way that we could simultaneously handle in the same framework L^2-integrable closed loop strategies and smooth locally bounded ones except at most in the neighborhood of finitely many instant of time as in [2].

It turns out that the classical definition of a closed loop saddle point can be an *undeterminate* or a *degenerate* one when either the open loop lower or upper value of the game is not finite. For instance, Berkovitz [1]'s equivalence Lemma may not apply as shown in an Example. The proper point of view of Definition is that the two closed loop strategies cannot be chosen independently. They must be linked through the *admissibility condition* of theDefinition. This subtle difference fundamentally changes the nature of the problem and makes it different from the classical theory of saddle points with respect to two independent sets. We show that the slight relaxation of the definition of *normalizability* of the free end problem in the sense of [2, Definition 3.2] from isolated instants to a set of instants of zero measure is a necessary and sufficient condition for the existence of a closed loop saddle point. This relaxation of the classical notion allows singularities in the feedback law at an infinite number of instants including accumulation points that are not isolated. This condition is also used to make sense of solutions with singularities to the matrix Riccati differential equation. We show that under the convexity-concavity condition, the definitions of closed loop saddle points coincide and that closed loop strategies with non L^2-integrable singularities are useless. They naturally occur when either the open loop lower or upper value of the game is not finite. We complete the classification of closed loop saddle points along with conditions expressed in terms of the convexity/concavity properties of the utility function. We conclude with an example of a non normalizable problem with finite open loop lower value that can be achieved by state feedback via a solution of the matrix Riccati differential equation.

References

[1] I. Berkovitz, *Lectures on Differential Games and Related Topics*, Kuhn and Szego, eds, North Holland Publishing Company, Amsterdam, Holland 1971.

[2] P. Bernhard, *Linear-quadratic, two-person, zero-sum differential games: necessary and sufficient conditions*, J. Optim. Theory Appl. 27 (1979), 51–69; *Technical comment to: "Linear-quadratic two-person zero-sum differential games: necessary and sufficient conditions" [J. Optim. Theory Appl. 27 (1979), no. 1, 51–69]*. J. Optim. Theory Appl. 31 (1980), 283–284.

[3] M. C. Delfour, *Linear-quadratic differential games: saddle point and Riccati differential equation*, SIAM J. Control Optim. 46, No. 2 (2007), 750–774.

[4] M. C. Delfour, and O. Dello Sbarba, *Linear-Quadratic Differential Games: Closed Loop Saddle Points*, SIAM J. Control Optim. 47, No. 6 (2009), pp. 3138–3166.

[5] P. Zhang, *Some results on two-person zero-sum linear quadratic differential games*, SIAM J on Control and Optim. 43, no. 6 (2005), 2157–2165.